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LETTER TO THE EDITOR

Current algebra, AKS theorem and new super evolution equations

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Abstract. We have deduced a new class of integrable super evolution equations by using a supersymmetric version of the AKS (Adler–Kostant–Sym) theorem in conjunction with the homogeneous space reduction technique of Marshall and current algebra of Samenov-Tian-Shansky.

In recent years Lie algebraic techniques have been extensively used to study the properties of nonlinear integrable evolution equations [1]. After the initial study, using Lie algebra to deduce the Bäcklund transformation [2] and an exact solution of integrable systems, there have been various ingenious variations to extend the integrable class. One such approach was the introduction of homogeneous space [3], and another was to use supersymmetric Lie algebra [4].

In this respect the same type of development took place in both cases of dynamical systems and evolution equations. In a recent communication Marshall [5] has shown that a homogeneous space reduction technique along with the use of the famous Adler–Kostant–Symes (AKS) theorem can lead to new integrable dynamical systems. Here in this communication we have generalised Marshall's approach in two ways. On the one hand we have considered a super-extension of the AKS theorem and on the other hand, instead of usual algebra, we have used the current algebraic structure with cocycles first put forward by Rieman–Tian–Shansky [6]. The super evolution equations that we generate are new.

Let g be a finite-dimensional Lie algebra [7] and

$$g = g_0 \oplus g_1 \tag{1}$$

be a $\mathbb{Z}_2$ graded decomposition of g, with a Lie bracket (super),

$$[X, Y] = XY - (-1)^{d(X)d(Y)} YX \tag{2}$$

where

$$d(X) = 0 \qquad \text{for } X \in g_0$$

$$d(X) = 1 \qquad \text{for } X \in g_1.$$

On g, we define a non-degenerate bilinear form $\langle\, ,\rangle$: some mapping $g \times g \to \mathbb{C}$, such that $\langle X, Y\rangle = 0$, if $X \in g_0$ and $Y \in g_1$ and

$$\langle X, Y\rangle = (-1)^{d(X)d(Y)} \langle Y, X\rangle.$$

On the other hand we can consider g to be of the form

$$g = \sum_{j \in \mathbb{Z}} g^{(j)}$$

and

$$X^{(j)} = X_j \lambda^j \in g^{(j)} \tag{3}$$

to be the elements of a loop algebra, whence we write

$$g = g^+ + g^-. \tag{4}$$

Let us consider now $g^* = g \oplus \mathbb{R}$, if $(x, a) \in g^*$ with $x \in g$ and $a \in \mathbb{R}$; the composition rule is given by [8]

$$[(x, a), (y, b)] = \left([x, y], \int \operatorname{str} XY' \, dx \right). \tag{5}$$

On g^* we define a degenerate bilinear form

$$\langle (x, a), (y, b) \rangle = ab + \int \operatorname{str} XY \, dx \tag{6}$$

with the property

$$\langle (x, a), (y, b) \rangle = (-1)^{d(X)d(Y)} \langle (y, b)(x, a) \rangle.$$

Now if we assume that g breaks up into affine Lie algebras of the form (4), then the pairing is defined by

$$\left\langle \left(\sum_j X_j \lambda^j, a(\lambda) \right), \left(\sum Y_k \lambda^k, b(\lambda) \right) \right\rangle$$
$$= \operatorname{res}\left(a(\lambda) b(\lambda) + \operatorname{str} \int \left(\sum X_j \lambda^j \right) \left(\sum Y_k \lambda^k \right) \right) dx. \tag{7}$$

Let us now consider a two-dimensional cocycle of g over $\mathbb{R}$ which is defined to be a smooth function f; such that [9]

$$f : \mathrm{G} \times \mathrm{G} \to \mathbb{R} \tag{8}$$

where G is the Lie group corresponding to the Lie algebra g. This function possesses the properties that

$$f(x, e) = f(e, x) = 0 \qquad e = \text{identity of G} \tag{9}$$

and

$$f(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0 \tag{10}$$

for any three $x, y, z \in \mathrm{G}$.

Now corresponding to the Lie algebra

$$g^* = g \oplus \mathbb{R}$$

we have the manifold M,

$$M = \mathrm{G} \otimes \mathbb{R}$$

where the multiplication of any two elements is defined as

$$(a, x)(b, y) = (a + b + f(x, y), xy) \tag{11}$$

for $\forall a$, $b \in \mathbb{R}$; x, y, $\in$G. The inverse is defined via

$$(a, x)^{-1} = (-a - f(x, x^{-1}), x^{-1}) \tag{12}$$

so that $(a, x)(a, x)^{-1} = (0, e)$. All elements of the form (a, e) constitute a subgroup A which is isomorphic to the group $\mathbb{R}$ and is in the centre of the group G.

We now analyse in detail the connection between the Lie algebra and Lie group defined above via exponential mappings.

$$\exp : g \to G.$$

If $a \in \mathbb{R}$ and $x \in g$ then

$$\exp(a, x) = (a, \exp x). \tag{13}$$

In particular we see that the one-parameter subgroup $t \to \beta_{(a,x)}(t)$ of M corresponding to an element (a, x) of h is given by

$$\beta_{(a,x)}(t) = (ta, \beta_x(t)). \tag{14}$$

Now from equation (11), we can deduce

$$[(a, x), (b, y)] = (W(x, y), x, y) \tag{15}$$

with $W(x, y) = f(x, y) - f(y, x)$.

The basics of nonlinear equations are embedded in the adjoint and co-adjoint of Lie groups over g. In general adjoint action of G over g is given by $Y \times Y^{-1}$ where $Y \in$ G and $X \in g$. Now on $M \equiv \mathbb{R} \times$ G, we get

$$\begin{aligned} \text{adj } gY &= \frac{\mathrm{d}}{\mathrm{d}t}[(ta, \mathrm{e}^{tx})(Y, 0)(-ta - f(\mathrm{e}^{tx}, \mathrm{e}^{-tx}), \mathrm{e}^{-tx})]_{t=0} \\ &= (f^0(x, y) - f^0(y, x), [x, y]) \end{aligned} \tag{16}$$

where, $f^0(x, y) = f(\exp X, \exp Y)$, X, $Y \in g$. So the adjoint action is defined via the cocycle term. In particular, as per our defining relations (11), we take the cocycle term to be

$$W(x, y) = \text{str} \oint XY' \, \mathrm{d}x'. \tag{17}$$

Now the operator ad^* of the co-adjoint action is determined via the equality

$$\begin{aligned} \langle ad^*(x, a)(u, c), (y, b)\rangle &= \langle (u, c), ad_{(x,a)}(y, b)\rangle \\ &= \left\langle (u, c), \left([X, Y], \int \text{str } XY' \, \mathrm{d}x\right)\right\rangle \end{aligned} \tag{18}$$

where the braces denote the scalar product defined in (6), whence we get

$$\left\langle (u, c), \left([x, y], \int \text{str } XY' \, \mathrm{d}x\right)\right\rangle = c \int \text{str } XY' + \text{str} \int (uxy - uyx) \, \mathrm{d}x \tag{19}$$

so immediately we get

$$\left\langle (u, c), \left([x, y], \int \text{str } xy' \, \mathrm{d}x\right)\right\rangle = \langle (-cx' + [u, x], 0), (y, b)\rangle \tag{20}$$

which is the basis of the current algebra approach to Lax equations as advocated by Reyman and Semenovtian-Shansky. The nonlinear equations are then determined via the Hamiltonians determined via the ad-invariant functions over the Lie group. It is to be noted that the decomposition given in (4) gives an orthogonal decomposition in the sense that

$$\hat{g} = K + N \qquad \text{where } N^* = K^{\perp}$$

and

$$K = \hat{g}^+ \qquad N = \hat{g}^- \qquad \text{with } [K, K^{\perp}] \in K^{\perp}.$$

Following Marshall, we now define subspace, $A_{r,s}$ via

$$A_{r,s} = \text{span}\left\{ \sum_{j=k}^{l} \mu_j \lambda^j \,\middle|\, \mu_j \in \hat{g}, \mu_j = 0 \quad \text{for } j < r \text{ or } j > s \right\}. \tag{21}$$

Then we have

$$N = \hat{g} \cap A_{-\infty} - 1$$
$$K = \hat{g} \cap A_{0,\infty}.$$

On the other hand for any Lie algebra $g = h + m$ such that h is an Abelian part then $[h, h] \subset h$; $[h, m] \subset m$.

As per the general construction of Adler, the ad-invariant form on the affine algebra is given as

$$\gamma = A\lambda^p + Q\lambda^{p-1} + W_{p-2} \tag{22}$$

with $Q \in m$, $W_{p-2} \in A_{0,p-2}$, then the Hamiltonian is given as

$$H(\gamma) = -\tfrac{1}{2}\text{str}(\gamma^2, \lambda^{-p+1}) \tag{23}$$

whence we get

$$\nabla H = -\gamma\lambda^{-p+1}. \tag{24}$$

So

$$\dot{\gamma} = -[\Pi_k(-\gamma\lambda^{-p+1}), \gamma] - (\Pi_k(-(\gamma\lambda^{-p+1}))_x)$$

where Π_k is the projection onto the K-space.

Let us denote

$$Z = \Pi_k(\gamma\lambda^{-p+1})$$
$$= \Pi_k(A\lambda + Q + W_{p-2}\lambda^{-p-1}) \tag{25}$$

so,

$$\dot{\gamma} = (A\lambda + Q)_x + [A\lambda + Q, \gamma]. \tag{26}$$

As a particular case we consider $p = 2$, so that

$$\gamma = A\lambda^2 + Q\lambda + W. \tag{27}$$

Let us now consider an element $A \in h$ and construct the subgroup:

$$t = C_g(A) = \{B \in g, [A, B] = 0\}.$$

The g can be written as $g = t + m$, so that

$$[t, t] \subset t \qquad [t, m] \subset m.$$

Let us take an element

$$\alpha = \lambda^2 A + \frac{\Lambda}{(n+1)^2} \qquad \text{with } \Lambda \in t \tag{28}$$

and evaluate

$$\gamma = B^{-1}\left[\lambda^2 A + \frac{\Lambda}{(n+1)^2}\right]B \tag{29}$$

with

$$B = (b_1\lambda^{-1}, e^{\beta_1\lambda^{-1}})(b_2\lambda^{-2}, e^{\beta_2\lambda^{-2}})(b_3\lambda^{-3}, e^{\beta_3\lambda^{-3}})$$

whence

$$\gamma = (C, e^{-\beta_3\lambda^{-3}}\, e^{-\beta_2\lambda^{-2}}\, e^{-\beta_1\lambda^{-1}})\left(\lambda^2 A + \frac{\Lambda}{(n+1)^2} e^{\beta_1\lambda^{-1}}\, e^{\beta_2\lambda^{-2}}\, e^{\beta_3\lambda^{-3}}\right)$$

$$= \left(C, \lambda^2 A + [A, \beta_1]\lambda + \left\{[A_1\beta_2] + \tfrac{1}{2}[[A, \beta_1], \beta_1] + \frac{\Lambda}{(n+1)^2}\right\}\right). \tag{30}$$

We then renew $[A, \beta_1] = Q$ and $P = [A, \beta_2] + \frac{1}{2}[Q, \beta_1]$. Now in the case of Su$(n+1)$-type Lie algebra

$$A = \mathrm{i}\,\mathrm{diag}(n, -1, -1, \ldots, -1)$$

and we denote the subspaces of Q corresponding to eigenvalues $\mathrm{i}(n+1)$ and $-\mathrm{i}(n+1)$ of ad$A$ as Q_+, and Q_-. We can then rewrite M as

$$\gamma = \lambda^2 A + Q\lambda + \left(P - \frac{1}{(n+1)}[Q_-, Q_+]\right) + \frac{\Lambda}{(n+1)^2}. \tag{31}$$

Substituting into equation (26), we get, by equating the coefficients of various powers of λ,

$$\lambda^2\colon [Q, A] + [A, Q] = 0$$

$$\lambda\colon [A, P] + \left[A, \frac{-1}{(n+1)}[Q_-, Q_+]\right] = 0. \tag{32}$$

Since $A \in t$; $[Q_-, Q_+] \in t$ and hence

$$[A, [Q_-, Q_+]] = 0$$

which, in dependent terms, yields

$$\dot{Q} = [A, P] \tag{33}$$

or we get

$$P = P_+ + P_- = \frac{-\mathrm{i}}{n+1}(\dot{Q}_+ - \dot{Q}_-)$$

where we write $Q = Q_+ + Q_-$ whence γ is finally written as

$$\gamma = \lambda^2 A + Q\lambda + \left\{\frac{-\mathrm{i}}{n+1}(\dot{Q}_+ - \dot{Q}_-) - \frac{\mathrm{i}}{n+1}[Q_-, Q_+] + \frac{\Lambda}{(n+1)^2}\right\}. \tag{34}$$

To give an explicit example of a nonlinear equation we now specialise our system to the simplest supergroup SL(2, 1), in which case Q may be written as

$$Q=\begin{pmatrix} 0 & a^* & \bar{f}_1 \\ -a & 0 & f_2 \\ -f_1 & -f_2 & 0 \end{pmatrix} \qquad \text{and} \qquad \Lambda=\begin{pmatrix} \Lambda_- & 0 & 0 \\ 0 & \Lambda_0 & 0 \\ 0 & 0 & \bar{\Lambda} \end{pmatrix}. \tag{35}$$

With this form of Q we now evaluate both sides of equation (26) whence we get

$$\begin{aligned} \ddot{a} &= \mathrm{i}(n+1)a_x - 2a|a|^2 - a\bar{f}_1 f_1 + a\bar{f}_2 f_2 + \bar{f}_2\dot{f}_1 + \dot{\bar{f}}_2 f_1 - \mathrm{i}(n+1)a(\Lambda_0 - \Lambda_-) \\ \ddot{f}_1 &= \mathrm{i}(n+1)f_{1x} - f_1|a|^2 + f_2\bar{f}_2 f_1 - \dot{a}f_2 + \dot{f}_2 a - \mathrm{i}(n+1)(\bar{\Lambda} - \Lambda_-)f_1 \\ \ddot{f}_2 &= \mathrm{i}(n+1)f_{2x} + f_2|a|^2 - (f_1\bar{f}_1)f_2 - \dot{a}^* f_1 - \dot{f}_1 a^* - \mathrm{i}\text{-}(n+1)(\bar{\Lambda} - \Lambda_0)f_2. \end{aligned} \tag{36}$$

In the above we denote the time derivatives by a dot, i.e. $\dot{f} = \partial f/\partial t$. These are an example of a new class of nonlinear integrable equations in two dimensions containing super variables.

In the analysis described above we have shown how the supersymmetric version of the AKS theorem can be used in conjunction with the homogeneous space reduction technique of Marshall to generate a coupled set of nonlinear integrable system which is supersymmetric.

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